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ON THE AVERAGE OSCILLATION OF A STACK

R. KEMP

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Evaluating a binary tree in postorder we assume that in one unit of time zero or two nodes are removed from the top of the stack and one node is stored in the stack. The oscillation of the stack can be described by a function f where $f(t)$ is the number of nodes in the stack after t units of time.

In this paper we shall first derive several new enumeration results concerning planted plane trees. In the second part we shall prove, that the average number of maxima (MAX-turns) and minima (MIN-turns) of the function f is $n/2$ and $n/2-1$, respectively, provided that all binary trees with n leaves are equally likely. Finally, we shall compute for large n and fixed j the average increase (decrease) of the length of the stack between the j-th MIN-turn and $(j+1)$-th MAX-turn (between the j-th MAX-turn and the j-th MIN-turn). This result implies that the average oscillation of the stack can be described by the function $f(j)=4\sqrt{j/\pi}-(-1)^j+O(1/\sqrt{j})$ for large n and fixed turn-number j.

1. Introduction

Let $T(n), n\in\mathbf{N}$, be the set of all *extended binary trees* ([11; p. 399]) with n leaves and let $T\in T(n)$. The *stack size* $S(T)$ is recursively defined by

$$S(T):= \textbf{IF } |T| = 1 \textbf{ THEN } 1 \textbf{ ELSE IF } S(T_1) > S(T_2)$$
$$\textbf{THEN } S(T_1) \textbf{ ELSE } S(T_2)+1;$$

where $|T|$ is the number of nodes of the tree T and $T_1(T_2)$ is the left (right) subtree of T. $S(T)$ is the maximum number of nodes stored in the stack during postorder-traversing of $T\in T(n)$ ([11; p. 316]). In [3] it is implicitly shown that the average stack size of a binary tree $T\in T(n)$ is asymptotically given by

$$S(n) = \sqrt{\pi n} -1/2+O(\ln(n)/\sqrt{n})$$

assuming that all trees with n leaves are equally likely. A generalization of this result to special classes of trees was published in [5] and [9].

Evaluating a tree $T\in T(n)$ in postorder we assume that in one unit of time zero or two nodes are removed from the top of the stack and one node is stored in

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the stack. In [10] it is shown that the average number of nodes stored in the stack after t units of time during postorder-traversing of $T\in T(n)$ is given by

$$R(n, t) = 4\sqrt{n\varrho(1-\varrho)/\pi}+O(n^{-1/2})$$

where $\varrho=t/2n$ is a constant and all trees with n leaves are equally likely.

Now, let $d_T(t)$ be the number of nodes in the stack after t units of time during postorder-traversing of $T\in T(n)$. We have $d_T(1)=d_T(2n-1)=1$, $d_T(m)=0$ for $m\geqq 2n$ and $|d_T(t+1)-d_T(t)|=1$ for $1\leqq t\leqq 2n-2$. To any $T\in T(n)$ we can assign the sequence of natural numbers $(d_T(1), d_T(2), \ldots, d_T(2n-1))$. Obviously, different trees have different sequences. This construction is clearly reversible, that is, each sequence $(a_1, a_2, \ldots, a_{2n-1})$ of natural numbers with $a_1=a_{2n-1}=1$ and $|a_{i+1}-a_i|=1$, $1\leqq i\leqq 2n-2$, corresponds to a tree $T\in T(n)$ with $d_T(i)=a_i$, $1\leqq i\leqq 2n-1$. Henceforth, we shall say, that the sequence $(d_T(1), d_T(2), \ldots, d_T(2n-1))$ is the *description* of $T\in T(n)$.

Now, let $T\in T(n)$ have the description $(d_T(1), d_T(2), \ldots, d_T(2n-1))$. The number $d_T(i)$, $i\in[2:2n-2]$, is said to be a *MAX-turn*, if $d_T(i-1)<d_T(i)$ and $d_T(i)>d_T(i+1)$. Similarly, the number $d_T(i)$, $i\in[2:2n-2]$, is said to be a *MIN-turn*, if $d_T(i-1)>d_T(i)$ and $d_T(i)<d_T(i+1)$. A *turn* is either a MIN-turn or a MAX-turn. Note, that a description with k MAX-turns has $(k-1)$ MIN-turns. Now, let $d_T(p_j)$, $1\leqq j\leqq k$, with $p_i\leqq p_{i+1}$, $1\leqq i\leqq k-1$, be the MAX-turns and $d_T(q_j)$, $1\leqq j\leqq k-1$, with $q_i\leqq q_{i+1}$, $1\leqq i\leqq k-2$, be the MIN-turns in the description $(d_T(1), d_T(2), \ldots, d_T(2n-1))$ of $T\in T(n)$. The sequence $(s_T(1), s_T(2), \ldots, s_T(2k-1))$ with $s_T(2j-1)=d_T(p_j)$, $1\leqq j\leqq k$, and $s_T(2j)=d_T(q_j)$, $1\leqq j\leqq k-1$, is called *stack-sequence* of $T\in T(n)$.

A tree $T\in T(n)$ is uniquely represented by its stack-sequence. Intuitively, the stack sequence $(s_T(1), s_T(2), \ldots, s_T(2k-1))$ of $T\in T(n)$ describes the oscillation of the stack during the evaluation of T in postorder; if j is even (odd), the length of the stack is increased (decreased) by $|s_T(j)-s_T(j+1)|$ between the j-th turn $s_T(j)$ and the $(j+1)$-th turn $s_T(j+1)$, $1\leqq j\leqq 2k-2$. An example of a tree $T\in T(7)$ is given in Figure 1. The following Table 1 summarizes the instantaneous configurations of the stack after t units of time during postorder-traversing of the tree T.

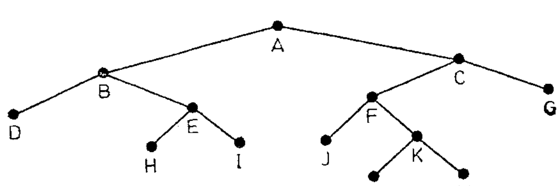

Fig. 1. A binary tree $T\in T(7)$

Table 1.

The nodes in the stack after t units of time. The rightmost node is the node on the top of the stack

t	1	2	3	4	5	6	7	8	9	10	11	12	13
nodes	*D*	*DH*	*DHI*	*DE*	*B*	*BJ*	*BJL*	*BJLM*	*BJK*	*BF*	*BFG*	*BC*	*A*

Obviously, T has the description (1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 3, 2, 1). Hence, the stack-sequence of T is (3, 1, 4, 2, 3). The oscillation of the stack can be illustrated by the graph given in Figure 2.

A *planted plane tree* is a rooted tree which has been embedded in the plane so that the relative order of subtrees at each branch is part of its structure. We shall use the convention that the leaves of a given planted plane tree are numbered 1, 2, 3, ... from left to right. The *level* of a node x is the number of nodes on the simple path from the root to node x including the root and node x. The *height* of a planted plane tree T is the maximum level of a node x appearing in T. Fixing two leaves of a planted plane tree T with number i and j, the tree $T_{i,j}$ is the uniquely determined subtree of minimal height with the leftmost leaf i and the rightmost leaf j (cf. Figure 3(b)).

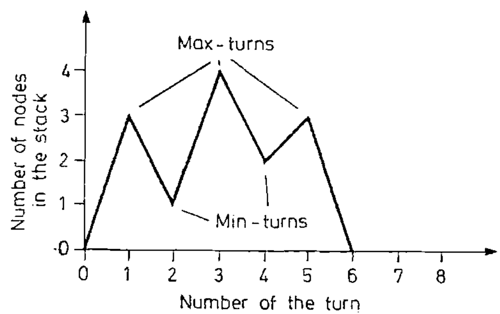

Fig. 2. The oscillation of the stack

In [3], [6], there is given a one-to-one correspondence between the planted plane trees with n nodes and the sequences $(a_1, a_2, \ldots, a_{2n-1})$ of natural numbers with $a_1 = a_{2n-1} = 1$ and $|a_{i+1} - a_i| = 1$, $1 \leqq i \leqq 2n-2$. Since the set of these sequences is identical to the set of all descriptions of all binary trees $T \in T(n)$, we can uniquely represent a tree $T \in T(n)$ by the corresponding planted plane tree T'. Thus, using the correspondence given in [3], the tree of Figure 1 can be represented by the planted plane tree of Figure 3(a).

Now, given $T \in T(n)$ with the stack-sequence $(s_T(1), s_T(2), \ldots, s_T(2k-1))$, the MIN-turns $s_T(2j)$, $1 \leqq j \leqq k-1$, and MAX-turns $s_T(2j-1)$, $1 \leqq j \leqq k$, can be illustrated in a natural way by means of the corresponding planted plane tree T'. Using the above one-to-one correspondence, it is easy to see, that the MAX-turn

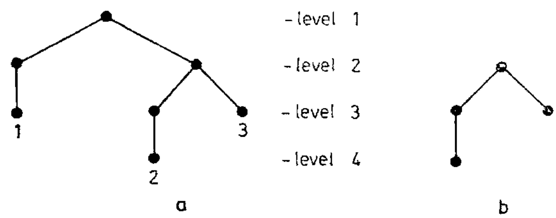

Fig. 3a. The corresponding planted plane tree T' of the binary tree T given in Figure 1

Fig. 3b. The tree $T'_{2,3}$

$s_T(2j-1), j\in[1:k]$, is equal to the level of the j-th leaf in the corresponding planted plane tree T'. Similarly, the MIN-turn $s_T(2j), j\in[1:k-1]$, is equal to the level of the root of the subtree $T'_{j,j+1}$ in the corresponding tree T'.

This paper is devoted to the "average oscillation" of the stack during postorder-traversing of a binary tree $T\in T(n)$. Considering all trees $T\in T(n)$ equally likely, we shall compute the average length of the stack after the j-th turn, where j is a fixed natural number. Moreover, we shall derive several new enumeration results describing the distribution of the number of certain binary trees. Throughout this paper we shall formulate our results in terms of planted plane trees.

2. Enumeration Results

We start our study of the enumeration of certain classes of planted plane trees by reviewing some known results. Let $t(n)$ be the number of planted plane trees with n nodes. It is well-known (e.g. [11; p. 389]) that $t(n)$ is the Catalan number

(1) $$t(n) = \frac{1}{n}\binom{2n-2}{n-1}$$

and that the generating function

(2) $$C(z) = \sum_{n\geqq 1} t(n)z^n$$

is given by

(3) $$C(z) = \frac{1}{2}(1-\sqrt{1-4z}).$$

Note, that $C^2(z)=C(z)-z$. The Taylor-coefficients of the powers of $C(z)$ are also well-known (e.g. [14; p. 154]). We have in general

(4) $$C^m(z) = \sum_{\mu\geqq 0}\left[\binom{2\mu+m-1}{\mu}-\binom{2\mu+m-1}{\mu-1}\right]z^{\mu+m}.$$

It is not hard to see, that the coefficient of z^{n-1} in the evaluation of $C^m(z)$ can be interpreted as the number of all n-node planted plane trees with a root of degree m ([9]).

Now, let $t(n,\lambda)$ be the number of planted plane trees with n nodes and λ leaves and let

$$F(z,y) = \sum_{n\geqq 1}\sum_{\lambda\geqq 1} t(n,\lambda)z^n y^\lambda$$

be the generating function of the numbers $t(n,\lambda)$. We obtain all planted plane trees with λ leaves by taking a root node and attaching zero or more subtrees where the total number of leaves of these subtrees is equal to λ. Hence in terms of the generating function $F(z,y)$

(5) $$F(z,y) = z\big(y+\sum_{r\geqq 1}F^r(z,y)\big) = zy+zF(z,y)/\big(1-F(z,y)\big).$$

Thus, $F(z,y)$ satisfies the equation $F^2(z,y)-\big(1+z(y-1)\big)F(z,y)+zy=0$. Solving this quadratic equation and using the fact $F(0,y)=0$ we obtain

(6) $$F(z,y) = \big(1+z(y-1)\big)C\big(zy/(1+z(y-1))^2\big)$$

where $C(z)$ is the generating function of the Catalan numbers given in (3). Now, we have to evaluate the function $F(z, y)$. In fact, we shall compute the coefficient of $z^n y^\lambda$ in the evaluation of $F^m(z, y)$ for $m \geqq 1$. It is easy to see, that this coefficient can be interpreted as the number of all $(n+1)$-node planted plane trees with a root of degree m and λ leaves.

Using (4), (6) and the well-known relation $(1-z)^{s+1} = \sum_{k\geqq 0} \binom{k+s}{k} z^k$ ([11; p. 90]) we obtain by an application of the binomial theorem

$$F^m(z,y) = \sum_{\mu\geqq 0}\left[\binom{2\mu+m-1}{\mu}-\binom{2\mu+m-1}{\mu-1}\right](zy)^{\mu+m}\big(1+z(y-1)\big)^{-2\mu-m} =$$

$$= \sum_{\mu\geqq 0}\sum_{k\geqq 0}\sum_{s\geqq 0}\left[\binom{2\mu+m-1}{\mu}-\binom{2\mu+m-1}{\mu-1}\right]\binom{2\mu+m-1+k}{k}\binom{k}{s}\times$$

$$\times(-1)^{k-s}(zy)^{\mu+m+k}y^{-s} = \sum_{n\geqq m}\sum_{\lambda\geqq m}(-1)^{\lambda-m}z^n y^\lambda \times$$

$$\times\sum_{\mu\geqq 0}\left[\binom{2\mu+m-1}{\mu}-\binom{2\mu+m-1}{\mu-1}\right]\binom{n+\mu-1}{n-\mu-m}\binom{n-\mu-m}{n-\lambda}(-1)^\mu =$$

$$= \frac{m}{n}\sum_{n\geqq m}\sum_{\lambda\geqq m}(-1)^{\lambda-m}\binom{n}{\lambda}z^n y^\lambda \sum_{\mu\geqq m}(-1)^{\mu-m}\binom{n-m-1+\mu}{n-1}\binom{\lambda}{\mu}.$$

Since $\binom{n-m-1+\mu}{n-1}=0$ for $\mu\leqq m-1$ we can use the general identity ([11; p. 58]) $\sum_{k\geqq 0}\binom{r}{k}\binom{s+k}{p}(-1)^k=(-1)^r\binom{s}{p-r}$ $r\geqq 0,\ p\geqq 0,\ s\geqq 0$ with $r:=\lambda$, $p:=n-1$ and $s:=n-m-1$. Hence

$$F^m(z, y) = z^m y^m + m\sum_{n\geqq m+1}\sum_{\lambda\geqq m} z^n y^\lambda \frac{1}{n}\binom{n}{\lambda}\binom{n-m-1}{\lambda-m}. \tag{7}$$

Since the coefficient of $z^n y^\lambda$ in $F^m(z, y)$ is the number $t_m(n+1, \lambda)$ of all $(n+1)$-node planted plane trees with a root of degree m and λ leaves, we have proved the following

Theorem 1. *The number $t_m(n, \lambda)$ of planted plane trees with n nodes, λ leaves and a root of degree m is given by*

$$t_m(n, \lambda) = \begin{cases} 1 & \text{if } n = m+1 \text{ and } \lambda = m \\ \dfrac{m}{n-1}\dbinom{n-1}{\lambda}\dbinom{n-m-2}{\lambda-m} & \text{if } n \geqq m+2 \text{ and } \lambda \geqq m \\ 0 & \text{otherwise.} \end{cases}$$ ∎

Since $t(n, \lambda)=t_1(n+1, \lambda)$ we have further the

Corollary 1. *The number $t(n, \lambda)$ of planted plane trees with n nodes and λ leaves is given by*

$$t(n,\lambda)=\begin{cases}1 & \text{if}\quad n=1 \quad \text{and} \quad \lambda=1\\ \dfrac{1}{n}\dbinom{n}{\lambda}\dbinom{n-2}{\lambda-1} & \text{if}\quad n\geqq 2 \quad \text{and} \quad \lambda\geqq 1\\ 0 & \text{otherwise.}\end{cases}$$ ∎

Returning to binary trees, Corollary 1 shows that there are $t(n, \lambda)$ binary trees with n leaves and λ MAX-turns. Using the one-to-one correspondence between binary trees and their descriptions, an alternative proof of this fact is implicitly given in [12; p. 19]. The numbers $t(n, \lambda)$ appear in the literature in connection with some other combinatorial problems (e.g. [12; p. 19], [13], [14; p. 17]).

For technical reasons, henceforth we shall constantly use the convention that the planted plane tree with one node has zero leaves, that is $t(1, \lambda)=\delta_{\lambda,0}$. If

$$\underline{F}(z, y)=\sum_{n\geqq 1}\sum_{\lambda\geqq 0} t(n, \lambda)z^n y^\lambda$$

is the generating function of the numbers $t(n, \lambda)$ with this convention, then we have by (6)

$$\underline{F}(z, y)=F(z, y)-zy+z=F(zy, y^{-1}). \tag{8}$$

Before we give some further enumeration results, we shall prove the following

Corollary 2. *Assuming that all planted plane trees with $n\geqq 2$ nodes are equally likely, the average number of leaves of such a tree is $n/2$. The variance is asymptotically given by $n(n-2)/(8n-12)$. An asymptotical equivalent for the s-th moment with respect to the origin is given by $(n/2)^s(1+O(n^{-1}))$.*

Proof. The quotient $p_\lambda(n)=t(n, \lambda)/(t(n)$ is the probability that a planted plane tree with n nodes has λ leaves. Using (1) and Theorem 1 we obtain for the s-th moment $m_s(n)$ with respect to the origin

$$m_s(n)=\sum_{\lambda\geqq 1}\lambda^s p_\lambda(n)=t^{-1}(n)\sum_{\lambda\geqq 1}\lambda^s t(n, \lambda)=t^{-1}(n)\frac{1}{n}\sum_{\lambda\geqq 1}\lambda^s\binom{n}{\lambda}\binom{n-2}{\lambda-1}.$$

Since in general ([1; p. 824])

$$\lambda^s=\sum_{0\leqq m\leqq s} m!\,\mathscr{S}_s^{(m)}\binom{\lambda}{m}$$

where the $\mathscr{S}_s^{(m)}$ are the Stirling numbers of the second kind we get further

$$m_s(n)=t^{-1}(n)\frac{1}{n}\sum_{0\leqq m\leqq s}\mathscr{S}_s^{(m)} m!\sum_{\lambda\geqq 1}\binom{\lambda}{m}\binom{n}{\lambda}\binom{n-2}{\lambda-1}.$$

Now, an application of the identity ([14; p. 15])

$$\binom{M}{p}\binom{N}{q}=\sum_{\lambda\geqq 0}\binom{M-N+q}{p+q-\lambda}\binom{\lambda}{q}\binom{N}{\lambda} \tag{9}$$

with $q:=m$; $p:=n-m-1$, $N:=n$ and $M:=2n-m-2$ leads to

$$m_s(n) = t^{-1}(n)\frac{1}{n}\sum_{0\leq m\leq s}\mathcal{S}_s^{(m)}m!\binom{n}{m}\binom{2n-m-2}{n-m-1}.$$

Using (1) and some special values for $\mathcal{S}_s^{(m)}$ ([1; p. 835]) we obtain immediately $m_1(n)=n/2$ and $m_2(n)=n(n^2-n-1)/(4n-6)$. The average number of leaves of a planted plane tree with n nodes is equal to the first moment $m_1(n)$. The variance is given by $\sigma^2(n)=m_2(n)-m_1^2(n)$. An elementary calculation leads to $\sigma^2(n)=$ $=n(n-2)/(8n-12)$, which completes the proof of the first part of our corollary.

Using Stirling's approximation we find immediately for fixed m

$$t^{-1}(n)\frac{m!}{n}\binom{n}{m}\binom{2n-m-2}{n-m-1} = (n/2)^m\left(1+O(n^{-1})\right)$$

and therefore

$$m_s(n) = \sum_{0\leq m\leq s}\mathcal{S}_s^{(m)}(n/2)^m\left(1+O(n^{-1})\right). \blacksquare$$

It is not hard to show, that the random variable X_n which assumes the value λ with probability $p_\lambda(n)$ is normally distributed with mean $m_1(n)$ and variance $\sigma^2(n)$. Returning to binary trees the above Corollary 2 shows, that in the average a given tree with n leaves has $n/2$ MAX-turns and therefore $(n-2)/2$ MIN-turns. This fact was also noted in [4]. We prove now the following

Theorem 2. *The number $t(n, j, i)$ of planted plane trees with $n\geq 2$ nodes, such that the j-th leaf has the level i is given by*

$$t(n,j,i)=\begin{cases}\binom{2n-i-2}{n-i}-\binom{2n-i-2}{n-i-1} & \text{if } j=1 \text{ and } 2\leq i\leq n\\ (i-1)\sum\limits_{0\leq\mu\leq n-i-1}\frac{1}{\mu+i}\binom{\mu+i}{j-1}\binom{\mu}{j-2}\left[\binom{2n-2\mu-i-4}{n-\mu-3}-\binom{2n-2\mu-i-4}{n-\mu-2}\right] & \\ \qquad\text{if } 2\leq j\leq n-1 \text{ and } 2\leq i\leq n-j+1 & \\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let for $i\geq 2$

(10) $$E_i(z,y) = \sum_{n\geq 1}\sum_{j\geq 1}z^n y^j t(n,j,i)$$

be the generating function of the numbers $t(n,j,i)$. Regarding Figure 4 we obtain all planted plane trees, such that the j-th leaf has the level i by

(a) taking the i-node tree of height i with the nodes $x_1, \ldots, x_i$, where x_1 is the root and x_i is the j-th leaf of the tree (giving the contribution $z^i y$)

and by

(b) attaching zero or more subtrees to the right of the nodes x_k, $1\leq k\leq i-1$, (giving the contribution $\left(z^{-1}C(z)\right)^{i-1}$) and

(c) attaching zero or more subtrees to the left of the nodes x_k, $1\leq k\leq i-1$, where the total number of the leaves of these subtrees is $j-1$ (giving the contribution $\left(z^{-1}\underline{F}(z,y)\right)^{i-1}$).

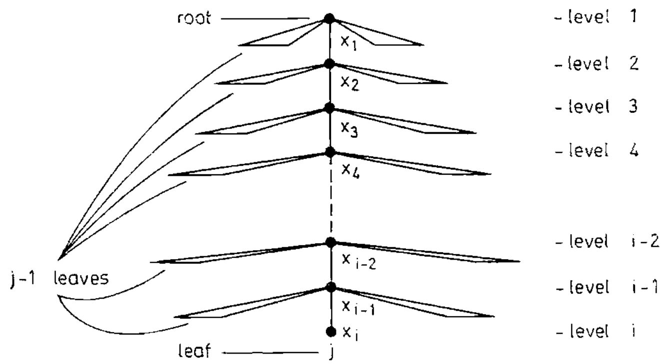

Fig. 4. The situation in the proof of Theorem 2

Thus

$$E_i(z, y) = z^{-(i-2)} y\{C(z)\underline{F}(z, y)\}^{i-1}$$

or with (8)

(11) $$E_i(z, y) = z^{-(i-2)} y\{C(z)F(zy, y^{-1})\}^{i-1}$$

where $C(z)$ and $F(z, y)$ are given in (3) and (6). Using (7), we obtain by a simple transformation

(12) $$F^{i-1}(zy, y^{-1}) = z^{i-1}+(i-1)z^i \sum_{n\geq 0}\sum_{\lambda\geq 1} z^n y^\lambda \frac{1}{n+i}\binom{n+i}{\lambda}\binom{n}{\lambda-1}.$$

A computation of the Cauchy product between $C^{i-1}(z)$ and $F^{i-1}(zy, y^{-1})$ leads with (11) to the explicit expression for $t(n, j, i)$ given in our Theorem 2. ▮

Returning to binary trees, Theorem 2 gives an explicit expression for the number of binary trees $T\in T(n)$, $n\geq 2$, which have exactly i nodes in the stack after the MAX-turn $s_T(2j-1)$.

Theorem 3. *The number $h(n, j, i)$ of planted plane trees with $n\geq 2$ nodes, such that the level of the root of the subtree $T_{j,j+1}$, $1\leq j\leq n-2$, is equal to $i\geq 1$ is given by*

$$h(n, j, i) = \begin{cases} \sum_{0\leq\mu\leq n-i-2}\binom{\mu}{j-1}\left[\frac{i}{\mu+i+1}\binom{\mu+i+1}{j} - \frac{i-1}{\mu+i}\binom{\mu+i}{j}\right]\times \\ \times\left[\binom{2n-2\mu-i-4}{n-\mu-2} - \binom{2n-2\mu-i-4}{n-\mu-1}\right] \\ \qquad \textit{if} \quad 1\leq j\leq n-2 \quad \textit{and} \quad 1\leq i\leq n-j-1 \\ 0 \qquad \textit{otherwise.} \end{cases}$$

Proof. Let for $i\geq 1$

(13) $$H_i(z, y) = \sum_{n\geq 1}\sum_{j\geq 1} z^n y^j h(n, j, i)$$

be the generating function of the numbers $h(n, j, i)$.

Regarding Figure 5 we obtain all planted plane trees such that the level of the root of the subtree $T_{j,j+1}$ (that is the node x_i) is equal to i and the j-th leaf has a level $k \geqq i+1$ by

(a) taking the k-node tree of height $k \geqq i+1$ with the nodes $x_1, ..., x_k$, where x_1 is the root and x_k is the j-th leaf of the tree (giving the contribution $z^k y$) and by

(b) attaching zero or more subtrees to the right of the nodes $x_1, ..., x_{i-1}$ (giving the contribution $(z^{-1}C(z))^{i-1}$) and

(c) attaching a subtree with at least two nodes to the right of the node x_i (giving the contribution $z^{-1}\,\mathrm{C}((z)-z)$) and

(d) attaching zero or more subtrees to the left of the nodes $x_1, ..., x_{k-1}$, where the total number of the leaves of these subtrees is $(j-1)$ (giving the conribution $(z^{-1}\underline{F}(z, y))^{k-1}$)

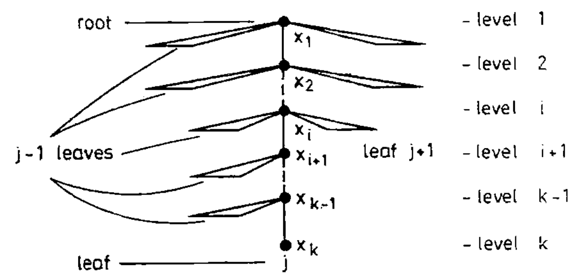

Fig. 5. The situation in the proof of Theorem 3

Since k ranges over $i+1, i+2, ...$ we obtain

$$H_i(z, y) = yz^{-(i-1)}C^{i-1}(z)(C(z)-z) \sum_{k \geqq i+1} \underline{F}^{k-1}(z, y).$$

Thus with (8)

$$H_i(z, y) = yz^{-(i-1)}\{C(z)F(zy, y^{-1})\}^{i-1}\{C(z)-z\}F(zy, y^{-1})/(1-F(zy, y^{-1})).$$

Since (5) implies

$$zyF(zy, y^{-1})/(1-F(zy, y^{-1})) = F(zy, y^{-1})-z$$

we get further

$$H_i(z, y) = z^{-i}\{C(z)\,F(zy, y^{-1})\}^{i-1}\{C(z)-z\}\{F(zy, y^{-1})-z\}$$

and therefore

$$H_i(z, y) = C^{i+1}(z)\{z^{-i}F^i(zy, y^{-1})-z^{-(i-1)}F^{i-1}(zy, y^{-1})\} \tag{14}$$

because $C^2(z) = C(z)-z$.

Using (12) and (4) we can compute the Cauchy products in (14) and obtain our explicit expression for $h(n, j, i)$ by a comparison of the coefficients. ∎

Returning to binary trees, Theorem 3 gives an explicit expression for the number of binary trees $T \in T(n)$, $n \geqq 2$, which have exactly i nodes in the stack after the MIN-turn $s_T(2j), j \geqq 1$.

3. On the average oscillation

Assuming that all binary trees T with n leaves are equally likely, this section is devoted to the computation of the average length of the stack after the j-th turn for fixed j. Thus, we have to compute the average level $\ell_n(j)$ of the j-th leaf (corresponds to the average number of nodes in the stack after the MAX-turn $s_T(2j-1)$) and the average level $r_n(j)$ of the root of the subtree $T'_{j,j+1}$ (corresponds to the average number of nodes in the stack after the MIN-turn $s_T(2j)$) in a planted plane tree T' with n nodes.

Using the notation of Theorem 2 and Theorem 3, we have by definition

(15a) $$\ell_n(j) = t^{-1}(n) \sum_{i \geqq 2} it(n, j, i)$$

and

(15b) $$r_n(j) = t^{-1}(n) \sum_{i \geqq 1} ih(n, j, i)$$

where $t(n)$ is given by (1). Now, let

(16a) $$\Delta_n(0) = \ell_n(1)$$

$$\Delta_n(j) = \ell_n(j+1) - r_n(j) \quad \text{for} \quad j \geqq 1$$

and

(16b) $$\nabla_n(j) = \ell_n(j) - r_n(j) \qquad \text{for} \quad j \geqq 1.$$

Returning to binary trees, $\Delta_n(j)$ is the average increase of the length of the stack between the MIN-turn $s_T(2j)$ and the MAX-turn $s_T(2j+1)$. Similarly, $\nabla_n(j)$ is the average decrease of the length of the stack between the MAX-turn $s_T(2j-1)$ and the MIN-turn $s_T(2j)$. Obviously, $r_n(j)$ and $\ell_n(j)$ can be expressed by the quantities $\Delta_n(j)$ and $\nabla_n(j)$. We obtain with (16a) and (16b)

(17a) $$\ell_n(j) = \Delta_n(0) + \sum_{1 \leqq \lambda \leqq j-1} [\Delta_n(\lambda) - \nabla_n(\lambda)]$$

and

(17b) $$r_n(j) = \Delta_n(0) - \nabla_n(j) + \sum_{1 \leqq \lambda \leqq j-1} [\Delta_n(\lambda) - \nabla_n(\lambda)].$$

Thus, it is sufficient to study the behaviour of $\Delta_n(j)$ and $\nabla_n(j)$. First, we prove the following

Lemma 1. *Let*

$$\Delta(z, y) = \sum_{n \geqq 1} \sum_{\lambda \geqq 0} z^n y^\lambda \Delta_n(\lambda) t(n)$$

be the generating function of the numbers $\Delta_n(\lambda)t(n)$. *We have*

$$\Delta(z, y) = \{1 + z(1-y)\} C^2(z) \{z(1-y)\}^{-1} - E_2(z, y) \{zy(1-y)\}^{-1}$$

where $C(z)$ *is given by* (3) *and* $E_2(z, y)$ *by* (11).

Proof. Let the functions $L(z, y)$ and $R(z, y)$ be defined by

(18a) $$L(z, y) = \sum_{i \geqq 2} iE_i(z, y)$$

and

(18b) $$R(z, y) = \sum_{i \geq 1} iH_i(z, y).$$

Using (15a) and (10), we get

(19a) $$L(z, y) = \sum_{n \geq 1} \sum_{\lambda \geq 1} z^n y^\lambda \ell_n(\lambda) t(n).$$

Similarly, with (15b) and (13),

(19b) $$R(z, y) = \sum_{n \geq 1} \sum_{\lambda \geq 1} z^n y^\lambda r_n(\lambda) t(n).$$

Thus,

$$\Delta(z, y) = y^{-1} L(z, y) - R(z, y).$$

On the other hand, an application of (11) and (14) leads to

(20a) $$L(z, y) = zy[1 - z^{-1} C(z) F(zy, y^{-1})]^{-2} - zy$$

and

(20b) $$R(z, y) = [z^{-1} C^2(z) F(zy, y^{-1}) - C^2(z)][1 - z^{-1} C(z) F(zy, y^{-1})]^{-2}.$$

Since $C^2(z) = C(z) - z$, the expression for $R(z, y)$ can be transformed into

(21) $$R(z, y) = z[1 - z^{-1} C(z) F(zy, y^{-1})]^{-2} - C(z)[1 - z^{-1} C(z) F(zy, y^{-1})]^{-1}.$$

Thus

$$\Delta(z, y) = C(z)[1 - z^{-1} C(z) F(zy, y^{-1})]^{-1} - z.$$

Now, using (3) and (6), a simple calculation shows, that

(22) $$[1 - z^{-1} C(z) F(zy, y^{-1})]^{-1} = 1 - [F(zy, y^{-1}) - C(z)][z(1 - y)]^{-1}.$$

With this relation we obtain further

$$\Delta(z, y) = C(z) - z - C(z)[F(zy, y^{-1}) - C(z)][z(1 - y)]^{-1} =$$
$$= C^2(z)[1 + z(1 - y)][z(1 - y)]^{-1} - E_2(z, y)[zy(1 - y)]^{-1}. \blacksquare$$

The following Lemma 2 can be proved in a similar way as the preceding Lemma 1. Using (19a), (19b), (20a), (20b) and (22) we obtain

Lemma 2. *Let*

$$\nabla(z, y) = \sum_{n \geq 1} \sum_{\lambda \geq 1} z^n y^\lambda \nabla_n(\lambda) t(n)$$

be the generating function of the numbers $\nabla_n(\lambda) t(n)$. We have

$$\nabla(z, y) = 2F(zy, y^{-1}) - C(z) - z - F^2(zy, y^{-1})[z(1 - y)]^{-1} + E_2(z, y)[zy(1 - y)]^{-1}. \blacksquare$$

Theorem 4.

(a) *We have* $\Delta_1(\lambda) = 0$ *for* $\lambda \geq 0$ *and for* $n \geq 2, j \geq 1$:

$$\Delta_n(0) t(n) = t(n+1)$$

$$\Delta_n(j) t(n) = t(n+1) - \sum_{1 \leq \lambda \leq j+1} t(n+1, \lambda, 2),$$

(b) *We have* $\nabla_1(\lambda)=0$ *for* $\lambda \geqq 0$ *and for* $n \geqq 2, j \geqq 1$:

$$\nabla_n(j)t(n) = 2t(n,j) - \sum_{1 \leqq \lambda \leqq j} t_2(n+2, n+1-\lambda) + \sum_{1 \leqq \lambda \leqq j+1} t(n+1, \lambda, 2).$$

The numbers $t(n)$, $t(n,j)$, $t_m(n,j)$ *and* $t(n,j,i)$ *are given by* (1), *Corollary* 1, *Theorem* 1 *and Theorem* 2.

Proof. (a) By definition, $\Delta_n(\lambda)t(n)$ is the coefficient of $z^n y^\lambda$ in the evaluation of $\Delta(z,y)$. Using Lemma 1, we obtain with (4) and (10) our proposition given in part (a).

(b) By definition, $\nabla_n(\lambda)t(n)$ is the coefficient of $z^n y^\lambda$ in the evaluation of $\nabla(z,y)$. Using Lemma 2, we obtain with (2), (10) and (12) for $n \geqq 2$ and $j \geqq 1$:

$$\nabla_n(j)t(n) = 2\frac{1}{n}\binom{n}{j}\binom{n-2}{j-1} - 2\frac{1}{n+1}\sum_{1 \leqq \lambda \leqq j}\binom{n+1}{\lambda}\binom{n-2}{\lambda-1} + \sum_{1 \leqq \lambda \leqq j+1} t(n+1, \lambda, 2).$$

An inspection of Corollary 1, Theorem 1 and Theorem 2 shows, that this expression is equivalent to part (b) of our Theorem. ▮

The following Theorem gives information about the behaviour of $\Delta_n(j)$ for large n and fixed j.

Theorem 5. (a) *The sequence* $\Delta_n(j)$ *is strictly decreasing in* j *for fixed* $n \geqq 2$ *and* $0 \leqq j \leqq n-2$

(b) $\Delta_n(j) = 0$ *for* $j \geqq n-1$

(c) $\Delta_n(0) = 4 + O(n^{-1})$

$$\Delta_n(j) = \frac{8}{3} - \frac{8}{9}\sum_{1 \leqq \lambda \leqq j-1} \frac{1}{\lambda(\lambda+1)} 3^{-\lambda} P'_\lambda(5/3) + O(n^{-1/2}) \quad \text{for fixed} \quad j \geqq 1.$$

Here, $P'_\lambda(5/3)$ *is the first derivative of the* λ*-th Legendre polynomial taken in the point 5/3.*

Proof. An inspection of Theorem 2 shows, that $t(n+1, \lambda, 2) > 0$ for $\lambda \leqq n$ and $t(n+1, \lambda, 2) = 0$ for $\lambda > n$.

(a) Using this fact we obtain immediately by Theorem 4(a)

$$\Delta_n(0) - \Delta_n(1) = \big(t(n+1, 1, 2) + t(n+1, 2, 2)\big)t^{-1}(n) > 0$$

and

$$\Delta_n(j) - \Delta_n(j+1) = t(n+1, j+2, 2)t^{-1}(n) > 0$$

for $1 \leqq j \leqq n-2$. This proves part (a).

(b) Using again the above fact we get with Theorem 4(a) in the case $j \geqq n-1$

$$\Delta_n(j)t(n) = t(n+1) - \sum_{1 \leqq \lambda \leqq n} t(n+1, \lambda, 2).$$

An application of Theorem 2 leads to

$$\sum_{1 \leqq \lambda \leqq n} t(n+1, \lambda, 2) = \binom{2n-2}{n-1} - \binom{2n-2}{n-2} +$$

$$+ \sum_{2 \leqq \lambda \leqq n} \sum_{0 \leqq \mu \leqq n-2} \frac{1}{\mu+2}\binom{\mu+2}{\lambda-1}\binom{\mu}{\lambda-2}\left[\binom{2n-2\mu-4}{n-\mu-2} - \binom{2n-2\mu-4}{n-\mu-1}\right].$$

Since always $\lambda-2\leqq\mu\leqq n-2$, we obtain further by (1), Corollary 1 and the identity given in (9) with $q:=0, p:=\mu+1, N:=\mu, M:=2\mu+2$

$$
\begin{aligned}
\sum_{1\leqq\lambda\leqq n} t(n+1,\lambda,2) &= t(n)+\sum_{0\leqq\mu\leqq n-2} t(n-\mu-1)\sum_{2\leqq\lambda\leqq\mu+2} t(\mu+2,\lambda-1) = \\
&= t(n)+\sum_{0\leqq\mu\leqq n-2} t(n-\mu-1)\,t(\mu+2) = \\
&= \sum_{0\leqq\mu\leqq n-1} t(n-\mu)\,t(\mu+1) = \\
&= t(n+1)
\end{aligned} \tag{23}
$$

because the last sum is equal to the coefficient of z^{n+1} in the evaluation of $C^2(z)$. This completes the proof of part (b).

(c) We define the numbers $D_n(j)$ for $n\geqq1$ and $j\geqq0$ by

$$D_n(j)=(j+1)\,t(n+2,j+2,2). \tag{24}$$

Using Theorem 4(a) and Theorem 2 we obtain immediately

$$[\Delta_{n+1}(0)-\Delta_{n+1}(1)]\,t(n+1)=t(n+1)+D_n(0) \tag{25a}$$

and

$$[\Delta_{n+1}(j)-\Delta_{n+1}(j+1)]\,t(n+1)=D_n(j)/(j+1) \tag{25b}$$

for $j\geqq1$.

Thus we have to study the asymptotic behaviour of $D_n(j)$ for fixed j and large n. An inspection of Theorem 2 shows, that

$$D_n(j)=\sum_{0\leqq\mu\leqq n-1}\frac{1}{n-\mu}\binom{2n-2\mu-2}{n-\mu-1}\binom{\mu}{j}\binom{\mu+1}{j}.$$

Therefore, $D_n(\lambda)$ is the coefficient of z^n in the evaluation of the function $\mathrm{D}_\lambda(z)=$ $=C(z)g_\lambda(z)$ where

$$g_\lambda(z)=\sum_{k\geqq0}\binom{k}{\lambda}\binom{k+1}{\lambda}z^k. \tag{26}$$

If $\lambda=0$, we get immediately $g_0(z)=(1-z)^{-1}$. Now, we regard $g_\lambda(z)$ for $\lambda\geqq1$. Since

$$2\binom{k}{\lambda}\binom{k+1}{\lambda}=\binom{k}{\lambda}^2+\binom{k+1}{\lambda}^2-\binom{k}{\lambda-1}^2$$

the sum $g_\lambda(z)$ can be transformed into

$$g_\lambda(z)=\frac{1}{2}[z^{-1}(1+z)Q_\lambda(z)-Q_{\lambda-1}(z)]$$

where

$$Q_\lambda(z)=\sum_{k\geqq0}\binom{k}{\lambda}^2 z^k.$$

Now, an elementary calculation leads to

$$Q_\lambda(z) = \frac{1}{\lambda!^2} z^\lambda \frac{d^\lambda}{dz^\lambda} z^\lambda \frac{d^\lambda}{dz^\lambda} \sum_{k \geqq 0} z^k = \sum_{0 \leqq r \leqq \lambda} \binom{\lambda}{r} \binom{\lambda+r}{\lambda} \frac{z^{\lambda+r}}{(1-z)^{\lambda+r+1}}.$$

Using Murphy's expression of the n-th Legendre polynomial ([15; p. 311]) given by

$$P_n(z) = \sum_{0 \leqq k \leqq n} \binom{n}{k} \binom{n+k}{k} 2^{-k}(z-1)^k$$

we obtain immediately

$$Q_\lambda(z) = \frac{z^\lambda}{(1-z)^{\lambda+1}} P_\lambda\left(\frac{1+z}{1-z}\right).$$

Thus for $\lambda \geqq 1$

$$g_\lambda(z) = \frac{z^{\lambda-1}}{2(1-z)^\lambda} \left[\frac{1+z}{1-z} \cdot P_\lambda\left(\frac{1+z}{1-z}\right) - P_{\lambda-1}\left(\frac{1+z}{1-z}\right)\right].$$

Generally, we have ([15; p. 309])

$$(z^2-1) \frac{d}{dz} P_n(z) = n\big(zP_n(z) - P_{n-1}(z)\big). \tag{27}$$

Hence for $\lambda \geqq 1$

$$g_\lambda(z) = \frac{2z^\lambda}{\lambda(1-z)^{\lambda+2}} P'_\lambda\left(\frac{1+z}{1-z}\right)$$

where $P'_\lambda\big((1+z)/(1-z)\big)$ stands for the first derivative of P_λ taken in the point $(1+z)/(1-z)$.

Altogether we have proved

$$D_\lambda(z) = C(z)g_\lambda(z) = \begin{cases} \dfrac{1-\sqrt{1-4z}}{2(1-z)} & \text{if } \lambda = 0 \\ \dfrac{z^\lambda(1-\sqrt{1-4z})}{\lambda(1-z)^{\lambda+2}} P'_\lambda\left(\dfrac{1+z}{1-z}\right) & \text{if } \lambda \geqq 1. \end{cases}$$

We are interested in the asymptotic behaviour of the coefficient $D_n(\lambda)$ of z^n in the evaluation of $D_\lambda(z)$. Using the Darboux-method ([2; p. 277]) we obtain for any $q \in \mathbf{N}$:

$$D_n(\lambda) = \begin{cases} \dfrac{4^n}{n!}\left[\dfrac{2}{3} \sum\limits_{0 \leqq p \leqq 2(q-1)} (-1)^{n+\lfloor (p+1)/2 \rfloor} 3^{-\lfloor p/2 \rfloor} (p/2)_n + O(n^{-q} n!)\right] & \text{if } \lambda = 0 \\ \dfrac{4^n}{n!}\left[\sum\limits_{0 \leqq p \leqq 2(q-1)} (-1)^{p+n} \dfrac{(p/2)_n}{\lfloor p/2 \rfloor !} \dfrac{d^{\lfloor p/2 \rfloor}}{dt^{\lfloor p/2 \rfloor}} f(t)\Big|_{t=0} + O(n^{-q} n!)\right] & \text{if } \lambda \geqq 1 \end{cases}$$

where $(x)_0 := 1$, $(x)_k := x(x-1) \cdot \cdot (x-k+1)$ for any $k \in \mathbf{N}$ and

$$f(t) = \frac{16(1-t)^\lambda}{\lambda(3+t)^{\lambda+2}} P'_\lambda\left(\frac{5-t}{3+t}\right).$$

Choosing $q=2$, a simple calculation shows, that the above expressions for $D_n(\lambda)$ are equivalent to

$$(29)\qquad D_n(\lambda)=\begin{cases}\dfrac{2}{3}\dfrac{1}{2n-1}\dbinom{2n}{n}+O(n^{-2}4^n) & \text{if } \lambda=0\\[2ex] \dfrac{16}{9\lambda}3^{-\lambda}P'_\lambda(5/3)\dfrac{1}{2n-1}\dbinom{2n}{n}+O(n^{-2}4^n) & \text{if } \lambda\geqq 1.\end{cases}$$

An inspection of Theorem 4(a) leads directly to

$$\Delta_n(0)=t(n+1)t^{-1}(n)=4+O(n^{-1}).$$

Since by Stirling's approximation

$$(30)\qquad t^{-1}(n+1)=n\sqrt{\pi n}\,4^{-n}(1+O(n^{-1}))$$

we obtain further with (25a) and the above approximation for $D_n(0)$:

$$\Delta_{n+1}(1)=\Delta_{n+1}(0)-1-t^{-1}(n+1)D_n(0)=\frac{8}{3}+O(n^{-1/2}).$$

Similarly, we get with (25b) and the above approximation for $D_n(\lambda)$, $\lambda\geqq 1$:

$$\Delta_{n+1}(\lambda)-\Delta_{n+1}(\lambda+1)=\frac{1}{\lambda+1}D_n(\lambda)t^{-1}(n+1)=\frac{8}{9}\frac{1}{\lambda(\lambda+1)}3^{-\lambda}P'_\lambda(5/3)+O(n^{-1/2}).$$

This equation implies immediately part (c) of our Theorem. ∎

Returning to binary trees, Theorem 5 shows, that in the average four nodes are in the stack, when the first MAX-turn is reached. The fact, that $\Delta_n(j)$ is a strictly decreasing in j for fixed $n\geqq 2$ means, that the average increase of the length of the stack between the MIN-turn $s_T(2j)$ and the MAX-turn $s_T(2j+1)$ is always less than the average increase of the length of the stack between the next MIN-turn $s_T(2j+2)$ and MAX-turn $s_T(2j+3)$. The first few values of the numbers $\Delta(j)=\lim_{n\to\infty}\Delta_n(j)$, j fixed, are summarized in Table 5.

Table 2.

The numbers $\Delta(j)=\lim_{n\to\infty}\Delta_n(j)$ and their approximation

j	0	1	2	3	4	5	6	7
$\Delta(j)$	4	2.6667	2.5185	2.4362	2.3832	2.3456	2.3172	2.2948
	—	2.7979	2.5642	2.4607	2.3989	2.3568	2.3257	2.3016

The following Corollary 5 gives information about the values $\Delta(j)$ for large j. The first few values of this approximation are given in the third line of Table 2.

Corollary 5. *We have:* $\Delta(j)=2[1+(2\pi j)^{-1/2}]+O(j^{-3/2})$.

Proof. We start from the integral representation of the Legendre polynomials ([7; p. 403)]

$$P_n(z) = \frac{1}{\pi}\int_0^\pi [z+\sqrt{z^2-1}\cos(\alpha)]^n\, d\alpha \tag{31}$$

for real $z \geqq 1$. Thus

$$P_n'(5/3) = \frac{3n}{4\pi}3^{-n}\int_0^\pi [4+5\cos(\alpha)][5+4\cos(\alpha)]^{n-1}\, d\alpha \tag{32}$$

and therefore

$$\sum_{1\leqq\lambda\leqq j-1}\frac{1}{\lambda(\lambda+1)}3^{-\lambda}P_\lambda'(5/3) = I(j)-R(j) \tag{33}$$

where

$$I(j) = \sum_{\lambda\geqq 1}\frac{1}{\lambda(\lambda+1)}3^{-\lambda}P_\lambda'(5/3) = \tag{34}$$

$$= -\frac{3}{4\pi}\int_0^\pi \frac{4+5\cos(\alpha)}{5+4\cos(\alpha)}\left[1+\frac{9}{5+4\cos(\alpha)}\ln\left(\frac{4}{9}(1-\cos(\alpha))\right)\right]d\alpha = \left(-\frac{3}{4\pi}\right)(-\pi) = \frac{3}{4}$$

and

$$R(j) = \sum_{\lambda\geqq j}\frac{1}{\lambda(\lambda+1)}3^{-\lambda}P_\lambda'(5/3).$$

Now, we shall compute the order of $R(j)$. We start with the asymptotic expansion of $P_n(z)$ for real $z\geqq 1$ given by ([7; p. 404])

$$P_n(z) = \frac{(z+z')^{n+1/2}}{\sqrt{2\pi n z'}}\left[1+\frac{z-2z'}{8z'n}+O(n^{-2})\right] \tag{35}$$

where $z'=(z^2-1)^{1/2}$. Using (27) we obtain by an elementary computation

$$P_n'(5/3) = \frac{9n}{8\sqrt{2\pi n}}3^n\left[1-\frac{7}{32n}+O(n^{-2})\right]$$

and therefore

$$\frac{1}{\lambda(\lambda+1)}3^{-\lambda}P_\lambda'(5/3) = \frac{9}{8\sqrt{2\pi}}\lambda^{-3/2}\left[1-\frac{39}{32\lambda}+O(\lambda^{-2})\right].$$

Hence

$$R(j) = \frac{9}{8\sqrt{2\pi}}\sum_{\lambda\geqq j}\lambda^{-3/2}-\frac{351}{256\sqrt{2\pi}}\sum_{\lambda\geqq j}\lambda^{-5/2}+\sum_{\lambda\geqq j}O(\lambda^{-7/2}).$$

The sums, appearing in the expression for $R(j)$ can be computed by means of Euler's summation formula. We obtain

$$R(j) = \frac{9}{4\sqrt{2\pi}}j^{-1/2}+O(j^{-3/2}). \tag{36}$$

Thus with Theorem 5(c)

$$\Delta(j) = \frac{8}{3}-\frac{8}{9}\big(I(j)-R(j)\big) = 2+2(2\pi j)^{-1/2}+O(j^{-3/2}). \quad \blacksquare$$

Next, we shall consider the behaviour of the numbers $\nabla_n(j)$ for fixed j and large n.

Theorem 6. (a) $\nabla_n(j)=0$ *for* $j \geqq n$,

(b) $\nabla_n(j)=\frac{4}{3}+\frac{8}{9}\sum_{1\leqq\lambda\leqq j-1}\frac{1}{\lambda(\lambda+1)}3^{-\lambda}P'_\lambda(5/3)+O(n^{-1/2})$ *for fixed* $j\geqq 1$.

Here, $P'_\lambda(5/3)$ *is the first derivative of the* λ*-th Legendre polynomial taken in the point* 5/3.

Proof. (a) An inspection of Theorem 2 shows, that $t(n+1, \lambda, 2)>0$ for $\lambda\leqq n$ and $t(n+1, \lambda, 2)=0$ for $\lambda\geqq n$. Similarly, we have by Theorem 1 and Corollary 1 that $t_2(n, \lambda)>0$ for $\lambda\leqq n-2$, $t_2(n, \lambda)=0$ for $\lambda>n-2$, $t(n, \lambda)>0$ for $\lambda\leqq n-1$ and $t(n, \lambda)=0$ for $\lambda>n-1$. Using these facts we get with Theorem 4(b) and (23) in the case $j\geqq n$

$$\nabla_n(j)\,t(n) = -\sum_{1\leqq\lambda\leqq n} t_2(n+2, n+1-\lambda)+\sum_{1\leqq\lambda\leqq n} t(n+1, \lambda, 2) =$$

$$= -t(n+1)+t(n+1) = 0$$

because by Theorem 1 and the identity given in (9) with $q:=0$, $p:=n-1$, $N:=n+1$, $M:=2n-1$

$$\sum_{1\leqq\lambda\leqq n} t_2(n+2, n+1-\lambda) = \frac{2}{n+1}\sum_{1\leqq\lambda\leqq n}\binom{n+1}{\lambda}\binom{n-2}{\lambda-1} = \frac{2}{n+1}\binom{2n-1}{n-1} = t(n+1).$$

(b) For $\lambda\geqq 1$, $n\geqq 1$, an application of Theorem 4(b) leads immediately to

$$[\nabla_{n+1}(\lambda)-\nabla_{n+1}(\lambda+1)]\,t(n+1) =$$

$$= 2t(n+1, \lambda)-2t(n+1, \lambda+1)+t_2(n+3, n+1-\lambda)-t(n+2, \lambda+2, 2).$$

Using the notation of $D_n(\lambda)$ in (24) and the explicit expressions for $t(n,j)$ and $t_m(n,j)$ given in Theorem 1 and Corollary 1, this equation can be easily transformed into

$$[\nabla_{n+1}(\lambda)-\nabla_{n+1}(\lambda+1)]\,t(n+1) =$$

$$= t_2(n+3, \lambda+1)-t(n+2, \lambda+2, 2) = -\frac{1}{\lambda+1}D_n(\lambda)+O(n^{2\lambda-1})$$

for fixed $\lambda\geqq 1$ and large n. Hence, by an application of (29) and (30)

$$\nabla_{n+1}(\lambda)-\nabla_{n+1}(\lambda+1) = -\frac{1}{\lambda+1}D_n(\lambda)t^{-1}(n+1)+O\big(n^{2\lambda-1}t^{-1}(n+1)\big) =$$

$$= -\frac{8}{9\lambda(\lambda+1)}3^{-\lambda}P'_\lambda(5/3)+O(n^{-1/2}).$$

This equation implies part (b) of our Theorem, provided that we can show that $\nabla_{n+1}(1)=4/3+O(n^{-1/2})$. We have by Theorem 4(b)

$$\nabla_{n+1}(1)\,t(n+1) = 2t(n+1, 1)-t_2(n+3, n+1)+t(n+2, 1, 2)+t(n+2, 2, 2).$$

An application of Theorem 1, Corollary 1, Theorem 2 and (24) leads to

$$\nabla_{n+1}(1) = 1+t^{-1}(n+1)D_n(0) = \frac{4}{3}+O(n^{-1/2})$$

if we use the approximation (29). ∎

A comparison of Theorem 5(c) with Theorem 6(b) shows, that the relation

$$\Delta_n(j)+\nabla_n(j) = 4+O(n^{-1/2})$$

holds for fixed $j \geqq 1$ and large n. Returning to binary trees, this fact means, that in the average the length of the stack is changed by four between two consecutive MAX-turns, provided that the number of the MAX-turns is independent of n. Furthermore, an inspection of Corollary 5 shows, that the oscillation of the stack is regular for large n and large $j<n$ where j is independent of n, that is, the stack is decreased or increased by two between consecutive turns. The average oscillation of the stack is illustrated in Figure 6.

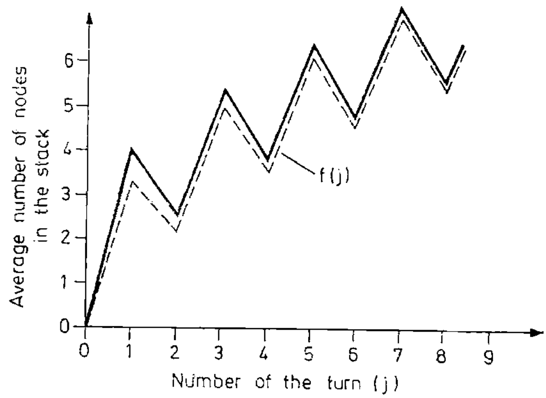

Fig. 6. The average oscillation of the stack for large n and fixed turn-number j

An inspection of (17a), (17b), Theorem 5(c) and Theorem 6(b) implies that for large n and fixed j the average level $\ell_n(j)$ of the j-th leaf and the average level $r_n(j)$ of the root of the subtree $T_{j,j+1}$ in a planted plane tree T with n nodes is given by

(37a) $$\ell_n(j) = \frac{4}{3}(j+2)-\frac{16}{9}S(j)+O(n^{-1/2})$$

and

(37b) $$r_n(j) = \frac{4}{3}(j+1)-\frac{8}{9}\sum_{1\leqq\lambda\leqq j-1}\frac{1}{\lambda(\lambda+1)}3^{-\lambda}P'_\lambda(5/3)-\frac{16}{9}S(j)+O(n^{-1/2})$$

where

$$S(j) = \sum_{1\leqq\lambda\leqq j-1}\sum_{1\leqq k\leqq\lambda-1}\frac{1}{k(k+1)}3^{-k}P'_k(5/3).$$

We prove the following

Corollary 6. *We have for large n and fixed j*

$$\text{(a)}\quad \ell_n(j) = 8\big(j/(2\pi)\big)^{1/2}+1+O(j^{-1/2}),$$

$$\text{(b)}\quad r_n(j) = 8\big(j/(2\pi)\big)^{1/2}-1+O(j^{-1/2}).$$

Proof. Obviously, the above expression for $S(j)$ is equivalent to

$$S(j) = j \sum_{1\le\lambda\le j-2} \frac{1}{\lambda(\lambda+1)} 3^{-\lambda}P'_\lambda(5/3)-Y(j)$$

where

$$Y(j) = \sum_{1\le\lambda\le j-2} \lambda^{-1}3^{-\lambda}P'_\lambda(5/3).$$

Making use of the integral representation of $P'_\lambda(5/3)$ given in (32) we obtain further

$$Y(j) = \frac{3}{16\pi}\int_0^\pi \frac{4+5\cos(\alpha)}{1-\cos(\alpha)}\left[1-\frac{5+4\cos(\alpha)}{9}\right]^{j-2} d\alpha =$$

$$= -\frac{15}{16}+\frac{27(j-2)}{4}3^{-(j-1)}P_{j-3}(5/3)+\frac{45}{16}3^{-(j-1)}P_{j-2}(5/3)+\frac{27}{4}3^{-(j-1)}P^1_{j-3}(5/3).$$

Here, we have used (31) and the integral representation of the associated Legendre function $P^m_\nu(z)$ given by ([15; p. 326])

$$P^m_\nu(z) = \frac{1}{\pi}\frac{(\nu+m)!}{\nu!}\int_0^\pi \left[z+\sqrt{z^2-1}\cos(\alpha)\right]^\nu \cos(m\alpha)\,d\alpha.$$

Thus, by Hobson's definition of the associated Legendre function ([15; p. 325]), that is

$$P^m_\nu(z) = (z^2-1)^{m/2}\frac{d^m}{dz^m}P_\nu(z),$$

we obtain further

$$Y(j) = -\frac{15}{16}+\frac{27}{16}3^{-j}[12(j-2)P_{j-3}(5/3)+5P_{j-2}(5/3)+16P'_{j-3}(5/3)].$$

Now, we can apply (27) and the asymptotic expansion of $P_n(z)$ for real $z>1$ given by (35). An elementary computation leads to

$$Y(j) = -\frac{15}{16}+\frac{9}{4}\big(j/(2\pi)\big)^{-1/2}+O(j^{-3/2}).$$

Hence, with (33), (34), (36) and this approximation, we find our result with (37a) and (37b). ∎

An inspection of the preceding corollary shows, that an approximation of the function which describes the oscillation of the stack is given by

$$f(j) = \frac{8}{\sqrt{2\pi}} \sqrt{\left\lfloor \frac{j+1}{2} \right\rfloor} - (-1)^j + O(j^{-1/2}).$$

Since $\lfloor (j+1)/2 \rfloor = (2j+1-(-1)^j)/4$, we can use the expansion of $(1+x)^{1/2}$ in this expression and obtain finally

$$f(j) = 4\sqrt{j/\pi} - (-1)^j + O(j^{-1/2}).$$

The oscillation described by this function is illustrated in Figure 6.

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R. Kemp

Johann Wolfgang Goethe Universität
Fachbereich Informatik (20)
D—6000 Frankfurt a.M.
F. R. G.